# Seminar Differential/Algebraic Topology

## Blakers-Massey, Freudenthal Suspension and Stable Homotopy Groups

Joshua P. Egger

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The **Freudenthal Suspension Theorem** is a fundamental result in homotopy theory thanks to its fundamental role in underpinning stable homotopy theory. The basic, and remarkable implication of the theorem is that

$$\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$$

for $i < 2n - 1$. The standard modern statement of the theorem deals with suspensions of general topological spaces, although Freudenthal's original result dealt only with spheres.

• In order to ensure that we're all on the same page, I will collect some basic concepts, so bear with me momentarily and don't feel insulted.

• **Def** ($n^{th}$ homotopy group): $X \in$ Top with basepoint $x_0 \in X$, $s \in S^n$ a chosen basepoint of the n-sphere. Then

$$\pi_n(X, x_0) = \{[f] : f : S^n \to X, f(s) = x_0\}$$

With group operation

$$(f+g)(t_1, ..., t_n) = \begin{cases} f(2t_1, ..., t_n) & t_1 \in [0, \frac{1}{2}] \\ g(2t_1 - 1, ..., t_n) & t_1 \in [\frac{1}{2}, 1] \end{cases}$$

• **Def**(*n-connected*) A space $X \in$ Top is called *n*-**connected** if $\pi_i(X) = 0$ for all $i \leq n$. 0-connected $\leftrightarrow$ path-connected, 1-connected $\leftrightarrow$ simply connected.

• Let $A \subseteq X$ and $x_0 \in A$. The $n^{th}$ **relative homotopy group** $\pi_n(X, A, x_0)$ is the collecton of homotopy classes of based maps $D^n \to X$ which take the boundary $\partial D^n = S^{n-1}$ to $A$. That is, for a given basepoint $s \in S^{n-1}$,

$$\pi_n(X, A, x_0) = \{[f] : f : D^n \to X, f(S^{n-1}) \subseteq A, f(s) = x_0\}$$

• An important advantage is that the relative homotopy groups fit into a long exact sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots$$

where $i_*$ and $j_*$ are the maps induced by the inclusions $A \hookrightarrow X$ and $(X, x_0) \hookrightarrow (X, A)$ respectively, and $\partial$ comes from restriction maps $(D^n, S^{n-1}) \to (X, A)$ to $S^{n-1}$

• Suppose $(X, A)$ is $n$-connected, so that $\pi_i(X, A) = 0$ for $i < n$. By the long exact sequence, this implies that the induced map $i_* : \pi_i(A) \to \pi_i(X)$ is an isomorphism for $i < n$ and a surjection for $i = n$. The inclusion $i : A \hookrightarrow X$ is called an **n-equivalence**.

• More generally, a map $f : X \to Y$ is called an n-equivalence if the induced map $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < n$ and a surjection for $i = n$. In this language, a weak homotopy equivalence is an $\infty$-equivalence.

- We now proceed with Blakers Massey.

**Def:** An **excisive triad** $(X, A, B)$ consists of a topological space $X$ along with two subspaces $A, B \subseteq X$ whose interiors cover $X$: $A^\circ \cup B^\circ = X$.

- The excision theorem in homology states that the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism on homology groups, and is one of the reasons why homology groups are so relatively easy to compute, e.g. the long Mayer Vietoris sequence of an excisive triad.

- Unfortunately the same does not hold in the context of homotopy groups: the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ does not in general induce an isomorphism on homotopy groups. One can come up with fairly simple counterexamples (e.g. a wedge product of two spheres).

- The Blakers-Massey excision theorem gives us conditions under which the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ for an excisive triad *does* yield an isomorphism on homotopy groups: it turns out that the homotopy groups will satisfy excision in a specific range of dimensions, a range which is roughly the sum of the connectivities of $(X, B)$ and $(A, A \cap B)$. We now state the theorem in a convenient form which will suffice for our purposes, but note that the result can be tightened up slightly (see the book by Tammo Tom Dieck, for example):

**Theorem** (Blakers-Massey): Let $(X, A, B)$ be an excisive triad such that the intersection $C = A \cap B$ is nonempty. Suppose $(A, C, *)$ is n-connected and $(B, C, *)$ is m-connected for each choice of basepoint $* \in C$. Then for every basepoint $* \in C$, the map

$$\pi_i(A, C) \to \pi_i(X, B)$$

induced by the inclusions is an isomorphism for $i < n + m$ and a surjection for $i = n + m$.

• Note that the conclusion is the same as saying that the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ is an $(n+m)$-equivalence.

• As our focus is squrely on the Freudenthal suspension theorem, we will only go through a hand wavey sketch of the proof. The strategy is to reduce an arbitrary excisive triad $(X, A, B)$ to something simpler. There are also more elegant proofs using the Hurewicz theorem and homotopy fibres which you may be interested in reading up on.

• First reduction: We prove Blakers-Massey for when $A$ is built from $C = A \cap B$ by attaching cells of dimension greater than $n$ and $B$ is built up from $C$ by attaching cells of dimension greater than $m$.

• Proof ‘sketch’: We claim that an $n$-connected pair $(A, C)$ can be replaced by another $n$-connected pair $(A', C)$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, hook] & A' \arrow[dl, "\cong"] \\ A & \end{tikzcd}$$

and $A'$ is built from $C$ by attaching cells of dimension greater than $n$ only. To see this, we build up a CW complex from $C$ by attaching cells which represent elements of $\pi_i(A)$ or get rid of elements which shouldn't be in $\pi_i(A)$. Since $\pi_i(C) \cong \pi_i(A)$ for all $i < n$, we only need to add cells of dimension greater than $n$ to make this work. The procedure can likewise be carried out for $(B, C)$ ‘$\Box$’

• Second Reduction: It suffices to prove Blakers-Massey when each of $A$ and $B$ is built from $C$ by attaching *one* cell apiece.

Proof sketch: By induction on the number of cells. We first claim that it's sufficient when $(A, C)$ has exactly one cell. Write $A = A' \cup e$ for $C \subseteq A' \subseteq A$, so that $(A, A')$ has one cell, and $(A', C)$ has exactly one cell less than $(A, C)$. Consider $X' = A' \cup_C B$, so that $X'$ is just $X$ without the cell $e$. The nugget of the proof is showing that if homotopy excision holds for the excisive triads $(X', A', B)$ and $(X, A, X')$ by induction, then it also holds for the triad $(X, A, B)$.

This follows via an application of the five lemma to the exact sequence of triples for both of the triples in the inclusion $(X', A', B) \hookrightarrow (X, A, X')$. We can then conclude that considering $(A, C)$ with one cell is sufficient.

The next step is to show that the same holds for $(B, C)$, and we follow a similar argument writing $B = B' \cup e'$, $X'' = A \cup_C B'$ for $C \subseteq B' \subseteq B$. If excision holds for the triads $(X', A, B')$ and $(X, X', B)$, it must also hold for $(X, A, B)$ since the inclusion

$$(A, C) \hookrightarrow (X, B)$$

can be factored as

$$(A, C) \to (X'', B') \to (X, B)$$

This concludes the proof of the second reduction.

We shown thus far that it suffices to consider $(X, A, B)$ with $A = C \cup e$ and $B = C \cup e'$ for single cells $e, e'$ of dimensions greater than $n$ and $m$ respectively. The core of the proof is to show that Blakers-Massey actually does hold for such triads. We'll go through a brief overview of this result, you can look in the literature for more detail.

**Lemma** Suppose that $X = A \cup_C B$ where $A = C \cup e$ and $B = C \cup e'$ are both built from $C$ by attaching cells of dimensions greater than $n$ and $m$ respectively. Then the map of relative homotopy groups $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < n + m$ and a surjection for $i = n + m$.

- Proof sketch: For any two interior points $x \in \mathring{e}$ and $y \in \mathring{e}'$ there is a diagram

$$\begin{tikzcd}
\pi_i(A, C) \arrow[r] \arrow[d, "\cong"'] & \pi_i(X, B) \arrow[d, "\cong"'] \\
\pi_i(X \setminus \{y\}, X \setminus \{x, y\}) \arrow[r] & \pi_i(X, X \setminus \{x\})
\end{tikzcd}$$

Where the fact that the vertical maps are isomorphisms follows from observin that $X \setminus \{x\}$ is homotopy equivalent to $B$ by retracting $e \setminus \{x\}$ to its boundary, and similar retractions give $X \setminus \{y\} \cong A$, $X \setminus \{x, y\} \cong C$.
Let's first do surjectivity. Take a representative of $\pi_i(X, B)$, which we consider here as a map $f : (I^i, \partial I^i) \to (X, B)$ taking $J^{n-1} := \mathrm{cl}(\partial I^n \setminus (I^{n-1} \times \{1\})$ (this is a standard and equivalent definition of the relative homotopy group) to the basepoint $* \in C$, i.e. $f$ maps the top face of $I^i$ into $B$, and the rest of the boundary to $*$.

Via the above diagram, it suffices to show tht $f$ is homotopic to a map $f'$ via a homotopy $h$, such that

- the image of $f'$ is in $X \setminus \{y\}$
- for every $t \in I$, the restriction of $h_t$ to the top face of $I^i$ avoids $x$
- for every $t \in I$, $h_t$ maps $J^{i-1}$ to $*$

If we can find such a map $f'$ and homotopy $h$, then we have shown that every representative in $\pi_i(X, B) \cong \pi_i(X, X \setminus \{x\})$ is homotopic to some representative in $\pi_i(X \setminus \{y\}, X \setminus \{x, y\})$. That is, $\pi_i(A, C) \to \pi_i(X, B)$ is surjective for $i \leq n + m$. I will leave out the proof that we can actually produce such maps $f'$ and $h$.

- Injectivity of $\pi_i(A, C) \to \pi_i(X, B)$ for $i < n + m$ follows via an almost identical argument: suppose we're given two representatives $g$ and $g'$ of $\pi_i(A, C)$ such that $[g] = [g'] \in \pi_i(X, B)$ via some homotopy $H : I^i \times I \to X$.

Replacing the map $f$ in the surjectivity argument with with $H$, our new claim is that we can find another homotopy $H'$ homotopic to $H$ via some homotopy $\Phi$ such that $H'$ misses $y$, the restriction of $\Phi_t$ to the top face of $I^{i+1}$ misses the point $x$, and $\Phi_t$ sends $J^i$ to the basepoint $*$.

This implies that there is a homotopy from $f$ to $g$ in $X \setminus \{y\}$ relative to $X \setminus \{x, y\}$. That is, $[g] = [g'] \in \pi_i(X \setminus \{y\}, X \setminus \{x, y\})$. This holds for $i + 1 \leq n + m$, as the domain of $H$ is $I^{i+1}$, while the domain of $f$ is $I^i$, whence $\pi_i(A, C) \to \pi_i(X, B)$ is injective.

The Blakers-Massey theorem does not directly help in computing homotopy groups, but it does allow us to relate the homotopy groups of spaces within a so-called 'stable' range.

• Let $(X, x_0)$ be a based space. The **suspension homomorphism** is the map $\Sigma_* : \pi_i(X) \to \pi_i(\Sigma X) : [f] \mapsto [\Sigma f]$ where

$$\Sigma f := f \wedge \mathrm{id}_{S^1} \to \Sigma X : [s, t] \mapsto [f(s), t]$$

We can now state the (generalised version of the) Freudenthal suspension theorem

**Theorem** (Freudenthal): Suppose $X$ is an $(n-1)$-connected based topological space. Then the suspension homomorphism $\Sigma_* : \pi_{i-1} \to \pi_i(\Sigma X)$ is an isomorphism for $i < 2n$ and a surjection for $i = 2n$. Succinctly stated, the suspension homomorphism is a $2n$-equivalence.

**Proof:** We'll prove the theorem by exhibiting a suitable excisive cover of the suspension $\Sigma X$, apply homotopy excision, and refer to the long exact sequences for some relative pairs.

Now the suspension $\Sigma X$ has a fairly natural excisive cover given by two reduced cones over $X$, one over and one under $X$, identified along their bases. Denote these cones by $Y_+$ and $Y_-$ with the intersection $Y_0 = Y_+ \cap Y_- = X \times \{\frac{1}{2}\}$, which is homotopy equivalent to $X$. Let's also denote a chosen basepoint $[x_0, \frac{1}{2}]$ in the intersection by just $x_0$. $Y_+$ and $Y_-$ are of course contractible onto $X \times \{0\}$ and $X \times \{1\}$ .

We now want to relate the homotopy groups of $X$ to the homotopy groups of the suspension. Let's first look at the long exact sequence for the pair $(\Sigma X, Y_\pm)$:

$$\cdots \to \pi_i(Y_\pm, x_0) \to \pi_i(\Sigma X, x_0) \to \pi_i(\Sigma X, Y_\pm, x_0) \to \pi_{i-1}(Y_\pm, x_0) \to \cdots$$

Plugging in what we already know, namely that $\pi_i(Y_\pm, x_0) = 0$, we obtain a short exact sequence

$$0 \to \pi_i(\Sigma X, x_0) \to \pi_i(\Sigma X, Y_\pm x_0) \to 0$$

whence $\pi_i(\Sigma X, x_0) \cong \pi_i(\Sigma X, Y_\pm, x_0)$. Likewise the LES for the pair $(Y_\pm, Y_0)$

$$\cdots \to \pi_i(Y_\pm, x_0) \to \pi_i(Y_\pm, Y_0, x_0) \to \pi_{i-1}(Y_0, x_0) \to \pi_{i-1}(Y_\pm, x_0) \to \cdots$$

gives us that $\pi_i(X, x_0) \cong \pi_{i+1}(Y_\pm, Y_0, x_0)$. Moreover, as $X$ is $(n-1)$-connected by hypothesis, i.e. $\pi_i(X, x_0) = 0$ for $i \leq n-1$, the pairs $(Y_\pm, Y_0)$ are $n$-connected, so upon application of Blakers-Massey we find that the map

$$i_* : \pi_i(Y_-, Y_0) \to \pi_i(\Sigma X, Y_+)$$

induced by the inclusion is an isomorphism for $i < 2n$ and a surjection for $i = 2n$.

We therefore have the diagram

$$\begin{tikzcd}
\pi_i(Y_-, Y_0) \arrow[r, "i_*"] \arrow[d, "\partial"', "\cong"] & \pi_i(\Sigma X, Y_+) \\
\pi_{i-1}(X) \arrow[r] & \pi_i(\Sigma X) \arrow[u, "j_*", "\cong"']
\end{tikzcd}$$

Where the first isomorphism is given by the boundary map $\partial : \pi_i(Y_-, Y_0, x_0) \to \pi_{i-1}(Y_0, x_0)$ and the second isomorphism by the inclusion $j_* : (\Sigma X, x_0, x_0) \hookrightarrow (\Sigma X, Y_+, x_0)$.
Everything we've done up to this point tells us that the bottom horizontal map is an isomorphism for $i < 2n$ and surjection for $i = 2n$, but it remains to be shown that this is, in fact, the suspension homomorphism $\Sigma_*$.

• In order to do this, we need to figure out where an element $[f] \in \pi_{i-1}(X)$ is sent to in $\pi_i(Y_-, Y_0)$ under the inverse of the boundary map $\partial^{-1}$. Given a representative $f : S^{i-1} \to X$, we want to find a map $g : (D^i, S^{i-1}) \to (Y_-, Y_0)$ such that the homotopy class of the restriction $\left[g|_{S^{i-1}}\right] = [f]$. A natural choice for such a map is

$$g : D^i, Y_- : t \cdot x \mapsto \left[f(x), \frac{t}{2}\right]$$

where $t \in [0,1]$ and $x \in S^{i-1}$. This map is continuous at $0 \in D^i$ and its restriction to $S^{i-1}$ is the composition

$$S^{i-1} \xrightarrow{f} X \xrightarrow{-, \frac{1}{2}} Y_0$$

so we do have that $[g] \in \pi_i(Y_-, Y_0)$, and therefore $\partial[g] = [f]$ and $\partial^{-1}[f] = [g]$

• Once we include into $\pi_i(\Sigma X, Y_+)$, there is enough wiggle room in $Y_+$ to find a homotopy from $g$ to the map

$$\hat{g} : D^i \to \Sigma X : t \cdot x \mapsto [f(x), t]$$

for example $(t \cdot x, s) \mapsto \left[f(x), \frac{t}{2-s}\right]$ does the job.

• Finally, we'd like to view $[\hat{g}]$ as an element of $\pi_i(\Sigma X)$ instead of as an element of $\pi_i(\Sigma X, Y_+)$. But as we discovered previously, the map $j_* : \pi_i(\Sigma X) \to \pi_i(\Sigma X, Y_+)$ is an isomorphism, induced by contracting $Y_+$ onto the basepoint $x_0$. But construction of the map $\hat{g}$, any boundary point $s \in S^{i-1}$ is sent to $[f(s), 1] = [x_0, 1] = x_0$, so $[\hat{g}]$ is in fact already an element of $\pi_i(\Sigma X)$, except that now we would like to see $\hat{g}$ as a map from $\Sigma S^i \cong S^{i+1}$, which we do by precomposing with the homeomorphism

$$\Sigma S^i \cong D^i / S^{i-1} : [x, t] \mapsto t \cdot x$$

The resulting map is then precisely the suspension of $f$. Collecting what we've just done, we have shown that the following diagram commutes

$$\begin{tikzcd}
{[g : t \cdot x \mapsto [f(x), t/2]]} \arrow[r] & {[g] = [\hat{g} : t \cdot x \mapsto [f(x), t]]} \arrow[d] \\
{[f : x \mapsto f(x)]} \arrow[u] \arrow[r] & {[\Sigma f : [x, t] \mapsto [f(x), t]]}
\end{tikzcd}$$

So the bottom horizontal map is indeed the suspension homomorphism, as we wanted, proving the theorem.

• I'll conclude by briefly defining stable homotopy groups. The Freudenthal suspension theorem as initially proven by Freudenthal was stated for $X = S^n$, and his goal was to calculate higher homotopy groups of spheres. Most saliently, it is true that

$$\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$$

for $i < 2n - 1$, and in particular, since we know that $\pi_1(S^1) \cong \pi_2(S^2) \cong \mathbb{Z}$, Freudenthal allows us to conclude that

$$pi_1(S^1) \cong \pi_2(S^2) \cong \cdots \cong \pi_n(S^n) \cong \cdots$$

Which motivated the study of stable homotopy groups. Given an arbitrary CW complex $X$, the Freudenthal suspension theorem combined with the fact that $\Sigma X$ is connected allows us to conclude that the $n^{th}$ suspension $\Sigma^n X = \Sigma(\Sigma^{n-1}(X))$, so the map

$$\Sigma_* : \pi_i(\Sigma^n X) \to \pi_{i+1}(\Sigma^{n+1} X)$$

is an isomorphism for $i < 2n - 1$. This implies that after some index $i$, the maps in the sequence

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+1}(\Sigma^2 X) \to \cdots \pi_{i+n}(\Sigma^n X) \to \cdots$$

eventually all become isomorphisms. That is, they eventually *stabilise*.

The $i^{th}$ **stable homotopy group of** $X$ is defined to be the colimit

$$\pi_i^s(X) := \mathrm{colim}_n(\pi_{i+n}(\Sigma^n X))$$